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Topological–algebraic properties of function spaces with set-open topologies ☆

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ABSTRACT

For a Tychonoff space X, we denote by $C_\lambda(X)$ the space of all real-valued continuous functions on X with set-open topology. In this paper, we study the topological–algebraic properties of $C_\lambda(X)$. Our main results state that (1) $C_\lambda(X)$ is a topological vector space (a topological group) iff λ is a family of C-compact sets and $C_\lambda(X) = C_{\lambda'}(X)$, where λ' consists of all C-compact subsets of every set of λ. In particular, if $C_\lambda(X)$ is a topological group, then the set-open topology coincides with the topology of uniform convergence on a family λ; (2) a topological group $C_\lambda(X)$ is ω-narrow iff λ is a family of metrizable compact subsets of X.

1. Introduction

The set $C(X)$ of all continuous real-valued functions on a Tychonoff space X has a number of natural topologies. We consider the following three topologies: the topology of uniform convergence on a family λ of subsets of the set X, the set-open topology, and the weak set-open topology.

The topology of uniform convergence is given by a base at each point $f \in C(X)$. This base consists of all sets $\{g \in C(X): \sup_{x\in X} |g(x) - f(x)| < \varepsilon\}$. The topology of uniform convergence on elements of a family λ (the λ-topology), where λ is a fixed family of non-empty subsets of the set X, is a natural generalization of this topology. All sets of the form $\{g \in C(X): \sup_{x\in F} |g(x) - f(x)| < \varepsilon\}$, where $F \in \lambda$ and $\varepsilon > 0$, form a base of the λ-topology at a point $f \in C(X)$.

The set-open topology on a family λ of nonempty subsets of the set X (the λ-open topology) is a generalization of the compact-open topology and of the topology of pointwise convergence. This topology was first introduced by Arens and Dugundji [1].

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All sets of the form $\{f \in C(X)\colon\ f(F) \subseteq U\}$, where $F \in \lambda$ and U is an open subset of real line $\mathbb{R}$, form a subbase of the λ-open topology.

A set $A \subseteq X$ is said to be bounded if $f(A)$ is a bounded subset of $\mathbb{R}$ for each $f \in C(X)$.

In 1970, in [3] Buchwalter introduced another natural topology on $C(X)$, the topology of uniform convergence on bounded subsets of X.

The weak set-open topology on a family λ of nonempty subsets of the set X (the λ^*-open topology) is a generalization of the topology of uniform convergence on bounded subsets of X and of the topology of pointwise convergence. All sets of the form $\{f \in C(X)\colon\ \overline{f(F)} \subseteq U\}$, where $F \in \lambda$ and U is an open subset of $\mathbb{R}$, form a subbase of the λ^*-open topology.

As a result, for a given family λ of subsets of the set X, the following three topologies arise on $C(X)$: the topology of uniform convergence on λ, the set-open topology, and the weak set-open topology.

Note that all three topology coincide, when λ consists of all finite (compact, countable compact, pseudocompact, sequentially compact, C-compact) subsets of X. Therefore $C(X)$ with the topology of pointwise convergence (compact-open, countable compact-open, pseudocompact-open, sequentially compact-open, C-compact-open topology) is a locally convex topological vector space. Since a weak set-open topology coincides with a topology of uniform convergence on λ, when λ consists of all bounded subsets X (see Theorem 3.2 in [7]), it follows that it is a locally convex topological vector space.

In the present paper we look at the properties of the family λ which imply that the space $C_\lambda(X)$ with the (weak) set-open topology is a topological group (topological vector space) under the usual operations of addition (and multiplication by scalars).

2. Main definitions and notation

In this paper, we consider the space $C(X)$ of all real-valued continuous functions defined on a Tychonoff space X. We denote by λ a family of non-empty subsets of the set X. We use the following notation for various topological spaces with the underlying set $C(X)$:

$C_\lambda(X)$ for the λ-open topology,
$C_{\lambda^*}(X)$ for the λ^*-open topology,
$C_{\lambda,u}(X)$ for the λ-topology.

The elements of the standard subbases of the λ-open topology, λ^*-open topology, and λ-topology will be denoted as follows:

$$[F, U] = \bigl\{f \in C(X)\colon\ f(F) \subseteq U\bigr\},$$
$$[F, U]^* = \bigl\{f \in C(X)\colon\ \overline{f(F)} \subseteq U\bigr\},$$
$$\langle f, F, \varepsilon\rangle = \Bigl\{g \in C(X)\colon\ \sup_{x\in F}\bigl|f(x) - g(x)\bigr| < \varepsilon\Bigr\},$$

where $F \in \lambda$, U is an open subset of $\mathbb{R}$ and $\varepsilon > 0$.

If X and Y are any two topological spaces with the same underlying set, then we use the notation $X = Y$, $X \leqslant Y$, and $X < Y$ to indicate, respectively, that X and Y have the same topology, that the topology on Y is finer than or equal to the topology on X, and that the topology on Y is strictly finer than the topology on X.

The closure of a set A will be denoted by $\overline{A}$; the symbol $\emptyset$ stands for the empty set. As usual, $f(A)$ and $f^{-1}(A)$ are the image and the complete preimage of the set A under the mapping f, respectively. The constant zero function defined on X is denoted by 0, more precisely by 0_X. We call it the constant zero function in $C(X)$.

We denote by $\mathbb{R}$ the real line with the natural topology.

We recall that a subset of X that is the complete preimage of zero for a certain function from $C(X)$ is called a zero-set. A subset O of a space X is called functionally open (or a cozero-set) if $X \setminus O$ is a zero-set. A family λ of non-empty subsets of a topological space (X, τ) is called a π-network for X if for any nonempty open set $U \in \tau$ there exists $A \in \lambda$ such that $A \subset U$. Throughout this paper, a family λ of nonempty subsets of the set X is a π-network. This condition is equivalent to the space $C_\lambda(X)$ being a Hausdorff space. The set-open topology (the weak set-open topology) does not change when λ is replaced with the finite unions of its elements. Therefore we assume that λ is closed under finite unions of its elements.

Recall that a subset A of a space X is called *C-compact subset* X (or $\mathbb{R}$-compact) if, for any real-valued function f continuous on X, the set $f(A)$ is compact in $\mathbb{R}$.

Note (see Theorem 3.9 in [10]) that the set A is a C-compact subset of X if and only if every countable functionally open (in X) cover of A has a finite subcover.

The remaining notation can be found in [4].

3. Main results

Lemma 3.1. *Let $C_\lambda(X)$ be topological group. Then the family λ consists of C-compact subsets X.*

Proof. Suppose that there is $A \in \lambda$ which is not C-compact. Then there is $f \in C(X)$ such that $f(A)$ is not compact. We can assume that $f(A)$ is not closed. Indeed, let $f(A)$ be closed and unbounded in $\mathbb{R}$. We take $h(t) = arctg(t)$. Then, $h(f(A))$ is not closed. Let us consider a point $a \in \overline{f(A)} \setminus f(A)$ and the subbasic open set $O(f) = [A, \mathbb{R} \setminus \{a\}]$ which contains the point $f \in C_\lambda(X)$.

Since $C_\lambda(X)$ is topological group, there is an open set $[B, (-\epsilon, \epsilon)]$ such that $0 \in [B, (-\epsilon, \epsilon)]$ and $f + [B, (-\epsilon, \epsilon)] \subset O(f)$. Choose $x_0 \in A$ such that $f(x_0) \in (a - \epsilon, a + \epsilon)$. Let $p = f(x_0) - a \in (-\epsilon, \epsilon)$ and let $g \equiv -p$ be a constant function. It is clear that $g \in C(X)$ and $g \in [B, (-\epsilon, \epsilon)]$. But $(f + g) \notin O(f)$, because $f(x_0) + g(x_0) = f(x_0) - p = a \notin \mathbb{R} \setminus \{a\}$. This contradicts our assumption that $f + [B, (-\epsilon, \epsilon)] \subset O(f)$. $\square$

Given a family λ of non-empty subsets of X, let $\lambda(C) = \{A \in \lambda$: for every C-compact subset B of the space X with $B \subset A$, the set $[B, U]$ is open in $C_\lambda(X)$ for any open set U of the space $\mathbb{R}\}$.

Lemma 3.2. *Let $C_\lambda(X)$ be topological group. Then $\lambda = \lambda(C)$.*

Proof. Suppose that $A \in \lambda$, $B \subset A$ and B is a C-compact subset X. We claim that $[B, U]$ is an open set in $C_\lambda(X)$ for each open set U in $\mathbb{R}$. Let $f \in [B, U]$.

Since $f(B)$ is a compact set and $f(B) \subseteq U$, there is $S_\epsilon(f(B)) = \{y\colon\ y \in \mathbb{R},\ \rho(y, f(B)) < \epsilon\}$ such that $S_\epsilon(f(B)) \subseteq U$. The set $W = f + [A, (-\epsilon, \epsilon)]$ is the open set in $C_\lambda(X)$. It remains to prove that $W \subset [B, U]$. If $g \in W$ and if $x \in B$, then $\rho(g(x), f(x)) = \rho(f(x) + h(x), f(x)) < \epsilon$, where $h \in [A, (-\epsilon, \epsilon)]$. It follows that $g(x) \in U$ and $W \subset [B, U]$. $\square$

Theorem 3.3. *For a space X, the following statements are equivalent.*

1. $C_\lambda(X) = C_{\lambda,u}(X)$.
2. *$C_\lambda(X)$ is a topological group.*
3. *$C_\lambda(X)$ is a topological vector space.*
4. *$C_\lambda(X)$ is a locally convex topological vector space.*
5. *λ is a family of C-compact sets and $\lambda = \lambda(C)$.*

Proof. (1) $\Leftrightarrow$ (5). By Theorem 1 and Theorem 2 in [9].

(5) $\Rightarrow$ (4). By Theorem 1.1 in [6], if λ is a family of bounded (C-compact) sets, then $C_{\lambda,u}(X)$ is a topological vector space. Now for each $A \in \lambda$, define p_A on $C(X)$ by $p_A(f) = sup\{|f(x)|\colon\ x \in A\}$. Also for each $A \in \lambda$ and $\epsilon > 0$, let $V_{A,\epsilon} = \{f \in C(X)\colon\ p_A(f) < \epsilon\}$. Let $\gamma = \{V_{A,\epsilon}\colon\ A \in \lambda,\ \epsilon > 0\}$. It can be easily shown that for each $f \in C(X)$, the sets of the family $f + \gamma = \{f + V\colon\ V \in \gamma\}$ form a neighborhood base at f. Since this topology is generated by a collection of seminorms, is it locally convex. Hence by (1), $C_\lambda(X)$ is a locally convex topological vector space.

(4) $\Rightarrow$ (3) $\Rightarrow$ (2). This is immediate.

(2) $\Rightarrow$ (5). By Lemmas 3.1 and 3.2. $\square$

Remark 3.4. In general, $C_\lambda(X) = C_{\lambda,u}(X)$ does not imply that the family λ is closed under C-compact subsets. Let, for instance, $X = \mathbb{R}$ and a family λ consists of all finite unions of segments $[a, b]$ where b, $a \in \mathbb{R}$ and $b > a$. Then $C_\lambda(\mathbb{R}) = C_{\lambda,u}(\mathbb{R})$ but $c \notin \lambda$ for every $c \in \mathbb{R}$.

Let $\bar{\lambda} = \{\overline{A}\colon\ A \in \lambda\}$. Note that the same weak set-open topology is obtained if λ is replaced by $\bar{\lambda}$. This is because for each $f \in C(X)$ we have $f(\overline{A}) \subseteq \overline{f(A)}$ and, hence, $\overline{f(\overline{A})} = \overline{f(A)}$. Consequently, $C_{\overline{\lambda^*}}(X) = C_{\lambda^*}(X)$. From now on, λ denotes a family of non-empty closed subsets of the set X.

Lemma 3.5. *Let $C_{\lambda^*}(X)$ be topological vector space. Then the family λ consists of bounded subsets X.*

Proof. Suppose that there is an unbounded set $A \in \lambda$. Then, there is $f \in C(X)$ such that $f(A)$ is unbounded. It follows that, for every $\alpha \in \mathbb{R} \setminus \{0\}$ and for every $\epsilon > 0$ we have $\alpha f(A) \not\subseteq (-\epsilon, \epsilon)$. The set $[A, (-\epsilon, \epsilon)]^*$ is an open neighborhood of 0_X in $C_{\lambda^*}(X)$. Let $O(f)$ be any neighborhood of f and $(-\gamma, \gamma)$ be any interval. Then $\alpha f \notin [A, (-\epsilon, \epsilon)]^*$ for every $\alpha \in (-\gamma, \gamma) \setminus \{0\}$, because $\alpha f(A) \not\subseteq (-\epsilon, \epsilon)$. Thus, the scalar multiplication cannot be continuous. $\square$

Given a family λ of non-empty subsets of X, let $\lambda(B) = \{A \in \lambda\colon$ for every bounded subset B of the space X with $B \subset A$, the set $[B, U]$ is open in the weak set-open topology on $C(X)$ for any open set U of the space $\mathbb{R}\}$.

Lemma 3.6. *Let $C_{\lambda^*}(X)$ be a topological group. Then $\lambda = \lambda(B)$.*

Proof. Let $A \in \lambda$, $B \subset A$. Then we claim that $[B, U]^*$ is an open set in $C_{\lambda^*}(X)$ for each open set U in $\mathbb{R}$. Let $f \in [B, U]^*$. Since A is a bounded set, the set $\overline{f(B)}$ is compact. There is $S_\epsilon(\overline{f(B)}) = \{y\colon\ y \in \mathbb{R},\ \rho(y, \overline{f(B)}) < \epsilon\}$ such that $S_\epsilon(\overline{f(B)}) \subseteq U$.

The set $W = f + [A, (-\epsilon, \epsilon)]^*$ is the open set in $C_{\lambda^*}(X)$. It remains to prove that $W \subset [B, U]^*$. If $g \in W$ and if $x \in B$, then $\rho(g(x), f(x)) = \rho(f(x) + h(x), f(x)) < \epsilon$, where $h \in [A, (-\epsilon, \epsilon)]^*$. It follows that $g(x) \in U$ and $W \subset [B, U]^*$. $\square$

Theorem 3.7. *For a space X, the following statements are equivalent.*

1. $C_{\lambda^*}(X) = C_{\lambda,u}(X)$.
2. $C_{\lambda^*}(X)$ *is a topological group.*
3. $C_{\lambda^*}(X)$ *is a topological vector space.*
4. $C_{\lambda^*}(X)$ *is a locally convex topological vector space.*
5. λ *is a family of bounded sets and* $\lambda = \lambda(B)$.

Proof. (1) $\Leftrightarrow$ (5). By Theorem 4.5 and Theorem 4.7 in [10].

(5) $\Rightarrow$ (4). By Theorem 1.1 in [6], $C_{\lambda,u}(X)$ is a locally convex topological vector space. Hence by (1), $C_{\lambda^*}(X)$ is a locally convex topological vector space.

(4) $\Rightarrow$ (3) $\Rightarrow$ (2). This is immediate.

(2) $\Rightarrow$ (5). By the Theorem 4.3 in [10], the family λ consists of bounded subsets X. Note that in proof of Theorem 4.3 in [10], we could have used only the fact that $C_{\lambda^*}(X)$ is a topological group. By Lemma 3.6, $\lambda = \lambda(B)$. $\square$

Note that (3) $\Rightarrow$ (5) by Lemmas 3.5 and 3.6.

4. τ-Narrow function spaces

There is a property of topological groups which is related to cellularity. Let G be an additive topological group, and let τ be an infinite cardinal number. Then G is τ-narrow provided that for each neighborhood of U of the identity in G, there exists a subset S of G such that $|S| \leqslant \tau$ and $G = \{s + u\colon\ s \in S \text{ and } u \in U\}$. Now G can be characterized as being τ-narrow if and only if it is isomorphic to a subgroup of a group of cellularity less than or equal to τ. Since $C_p(X)$ has the countable chain condition, it is always ω-narrow.

McCoy and Ntantu (see Theorem 4.2.6 in [8]) showed that the space $C_\lambda(X)$ is τ-narrow if and only if $w_\lambda(X) \leqslant \tau$, where $w_\lambda(X) = \sup\{w(A)\colon\ A \in \lambda\}$ and λ is a hereditarily closed, compact network on X.

The $\upsilon\lambda$-weight $w_{\upsilon\lambda}(X)$ of X is defined by $w_{\upsilon\lambda}(X) = \sup\{w(\overline{A}^{\upsilon X})\colon\ A \in \lambda\}$ where $\overline{A}^{\upsilon X}$ is the closure in Hewitt realcompactification υX of the set A. The next result gives a criterion for $C_\lambda(X)$ to be τ-narrow for any family λ.

Theorem 4.1. *A topological group $C_\lambda(X)$ is τ-narrow if and only if $w_{\upsilon\lambda}(X) \leqslant \tau$.*

Proof. Suppose that $C_\lambda(X)$ is τ-narrow. If $A \in \lambda$, then A is C-compact subset of X and $\overline{A}^{\upsilon X}$ is compact in Hewitt realcompactification υX. Let $\lambda_\upsilon = \{\overline{A}^{\upsilon X}\colon\ A \in \lambda\}$. It is easy to verify [5, p. 118] that the space $C_\lambda(X)$ is linearly homeomorphic to the space $C_{\lambda_\upsilon}(\upsilon X)$ (a function $g \in C(X)$ corresponds to the function $\tilde{g} \in C(\upsilon X)$ that is the continuous extension of g over υX). So $C(\upsilon X) = [\overline{A}^{\upsilon X}, (-\epsilon, \epsilon)] + S$ where $|S| \leqslant \tau$. Let $f = \Delta S$ be the diagonal product of the elements of S. Then f is a continuous mapping of υX to $\mathbb{R}^\tau$; therefore, $w(f(\upsilon X)) \leqslant \tau$. The restriction of f to $\overline{A}^{\upsilon X}$ is a continuous injection of $\overline{A}^{\upsilon X}$ to $\mathbb{R}^\tau$. Indeed, if x and y are different points of $\overline{A}^{\upsilon X}$, then there are disjoint zero-sets F and K, such that $x \in F$ and $y \in K$. Let h be a continuous mapping of υX to $\mathbb{R}$, such that $h[F] = \{0\}$ and $h[K] = \{3\epsilon\}$. So $h = p + s$ for some $p \in [\overline{A}^{\upsilon X}, (-\epsilon, \epsilon)]$ and $s \in S$. It follows that $s = h - p$ and $s(x) \neq s(y)$.

Since $\overline{A}^{\upsilon X}$ is compact, the restriction of f to $\overline{A}^{\upsilon X}$ is an embedding. Hence $w(\overline{A}^{\upsilon X}) \leqslant \tau$, so that $w_{\upsilon\lambda}(X) \leqslant \tau$.

For the converse, assume that $w_{\upsilon\lambda}(X) \leqslant \tau$ then for every $A \in \lambda$, $w(\overline{A}^{\upsilon X}) \leqslant \tau$. Let $[\overline{A}^{\upsilon X}, (-\epsilon, \epsilon)]$ be any basic neighborhood of $0_{\upsilon X}$. Consider the space $C_u(\overline{A}^{\upsilon X})$ endowed with the topology of uniform convergence on $\overline{A}^{\upsilon X}$. Since $w(\overline{A}^{\upsilon X}) \leqslant \tau$, the density of $C_u(\overline{A}^{\upsilon X})$ does not exceed τ (see Theorem 3.4.16 in [4]). Let $\epsilon > 0$ and let D be a dense subset of $C_u(\overline{A}^{\upsilon X})$ having cardinality $d(C_u(\overline{A}^{\upsilon X}))$. We conclude from Lemma 3.1 that A is a C-compact subset of X and hence $\overline{A}^{\upsilon X}$ is a compact and $\overline{A}^{\upsilon X}$ is C-embedded in υX. For every $d \in D$ there is an $\tilde{d} \in C(\upsilon X)$ such that $\tilde{d}|_{\overline{A}^{\upsilon X}} = d$. Let $S = \{\tilde{d}\colon\ d \in D\}$. For any $f \in C(\upsilon X)$ there is an $\tilde{d} \in S$ such that $\tilde{d} \in f + [\overline{A}^{\upsilon X}, (-\epsilon, \epsilon)]$ and hence $\tilde{d} = f + p$ for some $p \in [\overline{A}^{\upsilon X}, (-\epsilon, \epsilon)]$. It follows that $C(\upsilon X) = [\overline{A}^{\upsilon X}, (-\epsilon, \epsilon)] + S$ and hence $C_\lambda(X)$ is τ-narrow. $\square$

Theorem 4.2. *A topological group $C_{\lambda^*}(X)$ is τ-narrow if and only if $w_{\upsilon\lambda}(X) \leqslant \tau$.*

Proof. Note that if $A \in \lambda$, then A is bounded subset of X and $\overline{A}^{\upsilon X}$ is compact in Hewitt realcompactification υX. Now it is clear what modifications have to be made in the proof of Theorem 4.1 to finish the proof of this theorem. $\square$

Observe that we have the addition operation on the set $C(X)$ so the property of $C_\lambda(X)$ being τ-narrow can be considered without assuming that $C_\lambda(X)$ is a topological group.

If $C_\lambda(X)$ is any space (not necessarily topological group) then we obtain the following.

Theorem 4.3. *Let $C_\lambda(X)$ be τ-narrow. Then $w_\lambda(X) \leqslant \tau$.*

Proof. Let $A \in \lambda$. Then there exists a subset S of $C(X)$ such that $|S| \leqslant \tau$ and $C(X) = \{s+g\colon s \in S \text{ and } g \in [A, (-1/2, 1/2)]\}$. Let V be an open set in X and $x \in V \cap A$ then there is a continuous map of X onto $[0,1]$ such that $f(x) = 0$ and $f(X \setminus V) \subset \{1\}$. Since $f \in C(X)$, $f = g+s$ where $g \in [A, (-1/2, 1/2)]$ and $s \in S$. It is clear that $x \in s^{-1}(-1/2, 1/2)$ and $s^{-1}(-1/2, 1/2) \cap (A \setminus V) = \emptyset$. The family $\gamma = \{s^{-1}(-1/2, 1/2) \cap A\colon s \in S\}$ is a base in A and $|\gamma| \leqslant \tau$. □

Theorem 4.4. *Let $C_{\lambda^*}(X)$ be τ-narrow. Then $w_\lambda(X) \leqslant \tau$.*

The following counterexample (see Example 1 in [11]) shows that the converse of Theorem 4.3 need not be true.

Example 4.5 *(Velichko).* Let D be the discrete space having the cardinality $\mathfrak{c}$ of the continuum, and let $X = \overline{D}^{\omega}_{\beta D}$ be the ω-closure of D in the Stone–Čech extension βD (the union of the closures in βD of all countable subsets of D). The points of D and the closures of countable subsets of D form a base of a topology on X which has no subbase of smaller cardinality; thus, the weight of X equals $\mathfrak{c}$. It is easy to see that the space X is pseudocompact. Let λ be the set of all C-compact subsets of X. Then, for every $A \in \lambda$, $w(A) \leqslant w(X) = \mathfrak{c}$ and $w_\lambda(X) = \mathfrak{c}$. But $w_{\upsilon\lambda}(X) > \mathfrak{c}$ since $\overline{X}^{\upsilon X} = \beta D$.

We recall that $l(C_\lambda(X))$ and $c(C_\lambda(X))$ are the Lindelőf number and the cellularity of $C_\lambda(X)$, respectively.

Theorem 4.6. *Let $C_\lambda(X)$ be a topological group.*

1. *If $C_\lambda(X)$ satisfies $l(C_\lambda(X)) \leqslant \tau$, then $w_{\upsilon\lambda}(X) \leqslant \tau$ and $w_\lambda(X) \leqslant \tau$.*
2. *If $C_\lambda(X)$ satisfies $c(C_\lambda(X)) \leqslant \tau$, then $w_{\upsilon\lambda}(X) \leqslant \tau$ and $w_\lambda(X) \leqslant \tau$.*

Proof. The proof is obvious from the fact (see Proposition 5.1.13 in [2]) that if G is a topological group and $l(G) \leqslant \tau$ (or $c(G) \leqslant \tau$) then G is τ-narrow. □

Theorem 4.7. *A topological group $C_\lambda(X)$ is ω-narrow if and only if λ is a family of metrizable compact subsets of X.*

Proof. By Lemma 3.1, the family λ consists of, C-compact subsets X. By Theorem 4.3, for every $A \in \lambda$, $\gamma = \{s^{-1}(-1/2, 1/2) \cap A\colon s \in S\}$ is a countable base in A and $|\gamma| \leqslant \aleph_0$. Note that $\{s^{-1}(-1/2, 1/2)\colon s \in S\}$ is the family of functionally open sets of X. It follows that A is a metrizable compact subset of X. □

Corollary 4.8. *A space $C_\lambda(X)$ is ω-narrow topological group if and only if λ is a family of metrizable compact subsets of X and $\lambda = \lambda(C)$.*

Corollary 4.9. *A space $C_\lambda(X)$ is topologically isomorphic to a subgroup of a topological group having ccc if and only if λ is a family of metrizable compact subsets of X and $\lambda = \lambda(C)$.*

Corollary 4.10. *Let $C_\lambda(X)$ be a Lindelőf topological group. Then λ is a family of metrizable compact subsets of X and $\lambda = \lambda(C)$.*

Example 4.11 *(Dieudonné Plank).* Let ω_0, ω_1 denote respectively the first infinite and the first uncountable ordinal. Let X be $[0, \omega_1] \times [0, \omega_0] \setminus \{(\omega_1, \omega_0)\}$, the points of the deleted Tychonoff plank, with the topology τ generated by declaring open each point of $[0, \omega_1) \times [0, \omega_0)$, together with the sets $U_\alpha(\beta) = \{(\beta, \gamma)\colon \alpha < \gamma \leqslant \omega_0\}$ and $V_\alpha(\beta) = \{(\gamma, \beta)\colon \alpha < \gamma \leqslant \omega_1\}$.

Let $A = \{(\omega_1, n)\colon 0 \leqslant n < \omega_0\}$ and $Y = [0, \omega_1] \times [0, \omega_0]$. Let B be a nonempty C-compact subset of X. Since $\{\alpha\} \times [0, \omega_0]$ is clopen (and hence functionally open) for every $\alpha < \omega_1$, for every $\beta \leqslant \omega_0$, $([0, \omega_1] \times \{\beta\}) \cap B$ has at most a finite number of points. It follows that B is a compact subset of X.

In [7] it was proved that A is a closed bounded subset of X and $C_\lambda(X) < C_{\lambda_b^*}(X)$ where λ (λ_b) is the collection of all compact (bounded) subsets of X. Hence $C_\lambda(X) = C_{\lambda_{\mathbb{R}}}(X) < C_{\lambda_b^*}(X)$ where $\lambda_{\mathbb{R}}$ is the collection of all C-compact subsets of X. By Theorems 3.7 and 3.3, $C_{\lambda_b^*}(X)$ and $C_{\lambda_{\mathbb{R}}}(X)$ are topological groups, but $C_{\lambda_b}(X)$ is not a topological group and $C_{\lambda_{\mathbb{R}}}(X) < C_{\lambda_b}(X)$. Since X is C-embedded and dense in Y we can see that $\beta X = \beta Y$. It follows that $\upsilon X = \upsilon Y$ and $w_{\upsilon\lambda_b}(X) \leqslant \aleph_0$. Hence $C_{\lambda_b^*}(X)$ is ω-narrow topological group, but A is not a compact subset of X.

Note that since $w_{\upsilon\lambda_b}(X) \leqslant \aleph_0$ from the proof of Theorem 4.1 we have $C_{\lambda_b}(X)$ is ω-narrow (without assuming that $C_{\lambda_b}(X)$ is a topological group).

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